Final Exam

BX25203 Engineering Mathematics (I), Fall 2020
School of BioMedical Convergence Engineering, PNU
Dec. 9. 13:30 - 16:30

I. REMARK

- This is a closed book exam. You are permitted on three pages of notes.
- There are a total of 100 points in the exam. Each problem specifies its point total.
- You must SHOW YOUR WORK to get full credit.

II. PROBLEM SET

1) [10 points] Mark each statement True or False. You don't need to justify each answer in this problem (Let $\mathbb{P}_n$ be the set of polynomials where the degree is n.)
   a) $\mathbb{P}_2$ is the subspace of $\mathbb{P}_4$. [True/False]
   b) Consider the polynomials $\mathbf{p}_1(t) = 1 + t^2$ and $\mathbf{p}_2(t) = 1 - t^2$. Then, $\{\mathbf{p}_1, \mathbf{p}_2\}$ is a basis for $\mathbb{P}_3$. [True/False]
   c) Let A be an $n \times n$ matrix. Then, rank $A = n$ if and only if Nul $A = \{\mathbf{0}\}$. [True/False]
   d) Suppose $\mathbf{y}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$. Then, $\mathbf{y}$ is orthogonal to every $\mathbf{w}$ in Span $\{\mathbf{u}, \mathbf{v}\}$. [True/False]
   e) If A is an $m \times n$ matrix and the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some $\mathbf{b}$, then the columns of A span $\mathbb{R}^m$. [True/False]
   f) The columns of an invertible $n \times n$ matrix form a basis for $\mathbb{R}^n$. [True/False]
   g) $\det A^{-1} = (-1)\det A$ [True/False]
   h) The columns of $A \in \mathbb{R}^{n\times n}$ span $\mathbb{R}^n$ if and only if A has n pivot positions. [True/False]
   i) Assume A and B are $n \times n$ matrices. Then, $AB = \sum_{i=1}^{n} \text{col}_i(A)\text{row}_i(B)$. [True/False]
   j) If $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent, then $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are not in $\mathbb{R}^2$ [True/False].

2) [10 points] Let $\mathbb{P}_2$ be the set of polynomials where the degree of polynomials is 2. The set $B = \{1 + t^2, t + t^2, 1 + 2t + t^2\}$ is a basis for $\mathbb{P}_2$.
   a) Find the change-of-coordinates matrix from the basis B to the standard basis $E = \{1, t, t^2\}$.
   b) Find the change-of-coordinates matrix from the basis E to the standard basis B.
   c) Find the coordinate vector of $\mathbf{p}(t) = 1 + 4t + 7t^2$ relative to B.

3) [10 points] Diagonalize the matrix if possible.

$$A = \begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix}$$

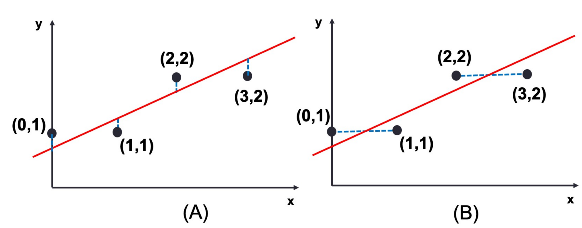
4) [10 points] Define T: $\mathbb{P}_2 \to \mathbb{R}^3$ by $\begin{bmatrix} \mathbf{p}(-1) \\ \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix}$
   a) Find the image under T of $\mathbf{p}(t) = 5 + 3t$.
   b) Show that T is a linear transformation.
   c) Find the matrix for T relative to the basis $\{1, t, t^2\}$ for $\mathbb{P}_2$ and the standard basis for $\mathbb{R}^3$.

5) [10 points] The linear equation is given as $A\mathbf{x} = \mathbf{y}$ where

$$A = \begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}.$$

   a) Solve the linear equation. Is the equation consistent?
   b) The matrix $U \in \mathbb{R}^{3\times 2}$ has two orthogonal columns. Find the U such that Col$A$ = Col$U$.
   c) Find the $\hat{\mathbf{y}} = \text{Proj}_{\text{Col}A}\mathbf{y}$ using the result of b).
   d) Find the solution of $\hat{\mathbf{y}} = A\mathbf{x}$. Explain why the equation must be consistent.
   e) Find the least-square solution using the normal equation. Check that it is same to the solution of d).

6) [10 points] The equation $y = \alpha_0 + \alpha_1 x$ is the line that best fits the given data points.



   a) Find the line of regression of y on x (Show (A)).
   b) Find the line of regression of x on y (Show (B)).

7) [10 points] For x and y in $\mathbb{P}_3$, define $< x, y >= x(-3)y(-3) + x(-1)y(-1) + x(1)y(1) + x(3)y(3)$. Let $p_0(t) = 1$, $p_1(t) = t$, and $p_2(t) = t^2$. Compute the orthogonal projection of $q(t) = t^3$ onto the subspace spanned by p_0, p_1 and p_2.

8) [10 points] Suppose x_k is the fraction of PNU students who like BTS more than linear algebra at year k. The

remaining fraction $y_k = 1 - x_k$ prefers linear algebra. Assume that

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix} \begin{bmatrix} x_k \\ y_k \end{bmatrix}, \quad \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

Note that all students (fraction=1) prefer BTS at $k = 0$. Explain what happens when $k \to \infty$.

9) [10 points] Prove that the orthogonal complement of $\mathrm{Row}A$ is $\mathrm{Nul}A$.

10) [10 points] Let the linear transformation be $T : \mathbb{P}_2 \to \mathbb{P}_2$. Assume that the linear transformation is invertible. $T(-4+2t+t^2) = 1$, $T(-t+t^2) = t$ and $T(3+3t) = t+t^2$. Then, fine T^{-1}. [Hint: Think of one of the problems in Midterm exam..]